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Technological Change
and Immigration

Francesc Dilmé

Adreça correspondència:
Departament de Teoria Econòmica
Facultat de Ciències Econòmiques i Empresarials
Universitat de Barcelona
Av. Diagonal 690,
08034 Barcelona, Spain
Email: f.dilme@lse.ac.uk

Abstract: We develop a simple model where two technologies are available to produce the same good, and we study under what conditions both will be used. We use the model to analyze the consequences of the simultaneous use of two different technologies for the economic variables and economic growth. Finally, we explore how migrations of factors affect the technological change and the performance of the economy.

JEL Classification: O11, O15, O33, O41.

Keywords: Technological change, Solow model, migration.

Resum: Desenvolupem un model simple on hi ha dues tecnologies disponibles per produir el mateix bé, i estudiem en quines condicions les dues tecnologies seran utilitzades. Utilitzem el model per analitzar les consequències de l'ús simultani de diferent tecnologies per les variables econòmiques i el creixement econòmic. Finalment, explorem com afecten les migracions de factors al canvi tecnològic i al comportament de l'economia.

Paraules clau: Canvi tecnològic, model de Solow, migració.

1 Introduction

After a new productive technology arises, different firms may use different technologies to produce the same product. Some times this fact can be interpreted as a transitory process, where the new technology is gradually adopted and finally is the only one that remains being used. Nevertheless, in other cases the two technologies may be used in the same industry during long periods of time.

Technological change has been introduced in a great variety of ways. Former theories of technological change, as the Solow model (1957), assume a continuous improvement of the technology through augmentation of the productivity of the factors. In models such those with expanding variety of products, introduced by Spence (1976) and Romer (1990), the technology changes along the time by discrete introductions of new productive factors. The Schumpeterian models of quality ladders, introduced by Schumpeter (1934) and Aghion and Howit (1992), assume discrete increases of the productivities of the different productive factors. Peretto and Seater (2007) develop a model where the factor shares of the production function, that is assumed to be always Cobb-Douglas over the time, can change through R&D investments performed by the firms. Givon (2006) presents a similar model using a CES production function. In all these models the technological change produces strictly better technologies that those that are being used. Then, in this kind of models, at every moment firms choose the last technology invented, and it is impossible to have persistent situations where two technologies are used.

In contrast, Parente and Prescott (2004), Hansen and Prescott (2002) and Kremer (1993), develop unified evolution models where two different technologies are available in every moment of the history. In this case, the evolution of the endowments of the economy along the history is crucial to determine which technology is used at every moment of time. These papers do not treat the problem of cohabitation specifically and in a sufficient depth, because the technological change takes place in a very short period of time.

The aim of this paper is twofold. First, we study the conditions that allow the simultaneous use of two technologies. Second, we analyze the effects of cohabitation on economic growth. To this end, we present a model that analyzes the equilibria of an economy with two production processes available and firms can choose which technology use to produce. We will consider the simple case where there are two factors, capital and labor, and the two production functions are neoclassic. The resulting aggregate production function is locally linear in the region where the two technologies are used.

When the factors can flow freely across technologies the conditions of cohabitation of two technologies imply that the factor payments are locally independent of the endowments of the economy. This fact allows us to analyze the effects of opening the economy to the international trade. If one factor payment is fixed exogenously, then only one technology will be used.

Moreover, we study how technological change is driven both by changes in the endowments and changes on the productivity of the different production factors. Increases in the productivity of the different factors alter the conditions of cohabitation, and then the outcome of the economy. We use this model to study the effects of the immigration when there is endogenous technological change. In particular, we analyze the redistribution of endowments across the different technologies in the economy. The effects of migration shocks are different depending on whether we consider adaption costs of the factors or not. The results are also different depending on whether one or two technologies are used.

This paper is divided in 5 sections. After this introduction, the second Section establishes the main assumptions of our model and analyzes their main implications, specially when there is cohabitation between both technologies. The third Section analyzes a growth model when the production function is the one obtained in Section 2. The fourth Section is a study of the implications of the immigration on the share of the resources among the two technologies. Finally, we present the conclusions of this paper. Additionally, there is an Appendix with the proofs of the main results.

2 The technology

We consider an economy that produces a unique good used both for consumption and investment. This good can be produced using two different technologies. Obviously, as there is an unique good, the price of this good will be independent of the technology used to produce it. The two technologies use two production factors, both available in our economy: labor and capital. The total labor force of the economy is denoted by L and the total capital stock is denoted by K. At every moment of time every unit of factor (both labor and capital) can only be used in one technology. Initially we will assume that both the capital and the labor are homogeneous and can flow across the technologies without adaption costs. This assumption will be relaxed in the section 4.

We assume that the two available technologies have neoclassical production functions. We denote the production function of the two different technologies by α and β. The production functions will be denoted by F_i, $i = \alpha, \beta$. We denote L_i and K_i as, respectively, the labor force and the capital stock that are endowed in the technology i at every moment of time. The production obtained using the technology i is given by

$$Y_i = F_i(K_i, L_i) = L_i.F_i\left(\frac{K_i}{L_i}, 1\right) = L_i.f_i(k_i) \ ,$$

where $k_i = K_i/L_i$, for $i = \alpha, \beta$. The markets clearing conditions are:

$$K_\alpha + K_\beta = K \quad \text{and}$$
$$L_\alpha + L_\beta = L \ .$$

In a competitive economy markets must clear. Using the definitions of k_α and k_β and the market clearing conditions, we obtain:

$$L_i = \frac{k_i - k}{k_i - k_j} \cdot L \quad \text{and} \quad K_i = \frac{k_i - k}{k_i - k_j} \cdot \frac{k_i}{k} \cdot K \ , \quad \text{for } i = \alpha, \beta.$$

We claim that the economy is under cohabitation when the quantities of productive factors endowed in both technologies are positive, that is, $K_i > 0$ and $L_i > 0$ for $i = \alpha, \beta$. Obviously, it is not necessary that the economy is under cohabitation, and only one technology may be used. When the economy is under cohabitation we have

$$k = \frac{L_\alpha}{L} \cdot \frac{K_\alpha}{L_\alpha} + \frac{L_\beta}{L} \cdot \frac{K_\beta}{L_\beta} = \frac{k_\beta - k}{k_\beta - k_\alpha} \cdot k_\alpha + \frac{k - k_\alpha}{k_\beta - k_\alpha} \cdot k_\beta \ , \tag{2.1}$$

where $k = \frac{K}{L}$. This formula says that k is a weighted average of the capital-labor ratios of the two technologies, with weights equal to the respective shares of the labor force, and thus $k \in (k_\alpha, k_\beta)$.[1] The aggregate production function is

$$y(k, k_\alpha, k_\beta) = \frac{k_\beta - k}{k_\beta - k_\alpha} \cdot f_\alpha(k_\alpha) + \frac{k - k_\alpha}{k_\beta - k_\alpha} \cdot f_\beta(k_\beta) \ . \tag{2.2}$$

We observe that the aggregate production function is a weighted average between the production function of the two technologies, with the same weights that appear in the formula of the capital-labor ratio (2.1).

Lets first treat the problem from the point of view of a single firm in our economy. Since both technologies have constant returns to scale, we assume that the firm has

[1]Henceforth, for simplicity, we will denote the set points between two reals x and y as (x, y), regardless the relative order of x and y.

$K^{\rm sf}$ unities of capital and tries to decide how many labor force $L^{\rm sf}$ should hire and how to share the resources between the technologies in order to maximize its profits. The firm takes the wage w and interest rate r of the economy as given. Then, the problem that faces the firm is[2]

$$\max_{K^{\rm sf}_\alpha, K^{\rm sf}_\beta, L^{\rm sf}_\alpha, L^{\rm sf}_\beta} \left[F_\alpha \left(K^{\rm sf}_\alpha, L^{\rm sf}_\alpha \right) + F_\beta \left(K^{\rm sf}_\beta, L^{\rm sf}_\beta \right) - w.(L^{\rm sf}_\alpha + L^{\rm sf}_\beta) \right] \ ,$$

$$\text{s.t.} \quad K^{\rm sf}_\alpha + K^{\rm sf}_\beta = K^{\rm sf}, \quad K^{\rm sf}_\alpha, K^{\rm sf}_\beta, L^{\rm sf}_\alpha, L^{\rm sf}_\beta \geq 0 \ .$$

Let's define, for each technology i, $k_i(w)$ as the capital-labor ration such that the marginal productivity of labor is exactly w. Since the two production functions are neoclassical this value exists for every $w > 0$. This is a Kuhn-Tucker maximization problem, with 5 restrictions, 4 of them may be non-binding. After doing some algebra, the system of first order conditions of this maximization problem can be rewritten in the following form

$$\left. \begin{array}{ll} k_i(w).L^{\rm sf}_i = K^{\rm sf}_i \ , & \text{for } i = \alpha, \beta, \quad \text{and} \\ \left[f'_i \left(k_i(w) \right) - f'_j \left(k_j(w) \right) \right] .K^{\rm sf}_i \geq 0 & \text{for } i, j = \alpha, \beta \ . \end{array} \right\} \tag{2.3}$$

The first equation says that the if the labor used in the technology i is different from 0, then it must be paid exactly w. This condition is natural, because the firm can choose the quantity of this production factor, and will hire new workers until their marginal production is w. The second condition says that some capital will be used in the technology i, and then $K^{\rm sf}_i > 0$, only when its marginal productivity is at least as high as the marginal productivity of the capital endowed in the other technology.

Note that, since w is given, $k_\alpha(w)$ and $k_\beta(w)$ also are given. Then, in general, the marginal productivities of the capital will be different for the two technologies in these values. This will generate a border solution, where only will be used the technology with high marginal productivity of the capital when the marginal productivity of the labor is w. Nevertheless, there may exist some values for the market wage such that the marginal productivity of the capital is the same in both technologies and the productivity of the labor coincides with the market wage. Assume that w_c verifies these properties. Then, the firm is indifferent to choose any combination

[2] The fact that the firm is small implies that it takes the rental price of capital as given. Then, both if the capital is owned by the firm or if it is rented, the firm considers this (opportunity) cost as a sunk cost, and not takes it in consideration when maximizes the production.

of labor and capital among technologies such that

$$L_\alpha^{\rm sf}(K_\alpha^{\rm sf}) = \frac{K_\alpha^{\rm sf}}{k_\alpha(w)} \quad \text{and} \quad L_\beta^{\rm sf}(K_\alpha^{\rm sf}) = \frac{K^{\rm sf} - K_\alpha^{\rm sf}}{k_\beta(w)} \ .$$

Then, the firm is indifferent to choose any $K_\alpha^{\rm sf} \in [0, K^{\rm sf}]$, because hiring the corresponding labor forces $L_\alpha^{\rm sf}(K_\alpha^{\rm sf})$ and $L_\beta^{\rm sf}(K_\alpha^{\rm sf})$ faces profits equal to 0. Note that, in this case, the border solutions, 0 and $K^{\rm sf}$, are also included in the range of possible values of $K_\alpha^{\rm sf}$. Note that, in this case, the capital-labor ratio of the firm must lay in the interval $[k_\alpha(w), k_\beta(w)]$.

A consequence of this result is that for some values of w (or R) the capital-labor ratio used by the firms may be different. This is an important difference of the case when two technologies are available with respect the case where only one technology is available. When only one technology is available the firms are scale-identical, in the sense that all use the same capital-labor ratio and only differ on the size of the firm. In our case, when there is cohabitation, the firms can differ both on the scale and on the capital-labor ratio, fact that introduces a deeper firm heterogeneity.

Consider now the economy at an aggregate level. Let's now introduce the competitive market, where the wages and the capital rents will be determined by the marginal product of the labor and the capital, respectively. We first inquire about the conditions that allow the economy to be under cohabitation. When only one technology is used then the classical analysis of a neoclassical production function applies.

When the economy is under cohabitation, factors mobility across the technologies implies that wages and capital rents must be equal in both technologies. The equation for the equalization of the wages between the two technologies is

$$f_\alpha(k_\alpha) - k_\alpha . f'_\alpha(k_\alpha) = w_\alpha = w_\beta = f_\beta(k_\beta) - k_\beta . f'_\beta(k_\beta) \ . \tag{2.4}$$

Assuming the same depreciation rate of the capital, the rental prices in both sectors will be the same. Then, the corresponding equation for the rental prices is:

$$f'_\alpha(k_\alpha) = R_\alpha = R_\beta = f'_\beta(k_\beta) \ . \tag{2.5}$$

The equations (2.4) and (2.5) generate a system of 2 equations and 2 unknown variables, k_α and k_β. In general the solution of this system is a (numerable) set of pairs of solutions for k_α and k_β, that we will denote $\big\{\{k_{\alpha c}^s, k_{\beta c}^s\},\ s = 1, \ldots\big\}$. Recall that, due the fact that neither (2.4) nor (2.5) depend on k, the set of solutions does not depend on k. Then, we can denote as R_c^s and w_c^s as the rental price and wage associated to the pair $\{k^s_{\alpha c}, k^s_{\beta c}\}$, for $s = 1, ...$, that, by assumption, are equal in both technologies. Using the equations (2.4) and (2.5) in the equation system (2.3) ensure that, under cohabitation, firms are indifferent to choose one technology or the other one.[3]

Let's now inquire about the conditions that allow the cohabitation, introducing some mathematical results. The next result, expressed as a proposition, says that the intervals of $\mathbb{R}$ with pairs of solutions as boundaries do not intersect. Then, the solutions of (2.4)-(2.5) generate a set of disjoint intervals of $\mathbb{R}$, that, as we will later see, will be the regions where cohabitation is be possible. The demonstration of the propositions and theorems are shown in the Appendix of this paper.

Proposition 1: *The intervals generated by the pairs of solutions of (2.4)-(2.5) do not intersect.*

This proposition allows us to analyze separately what occurs inside the intervals $(k^s_{\alpha c}, k^s_{\beta c})$, for $s = 1, ...$, and outside them. This is due to the fact that, as we have just said, one capital-labor ratio k can belong to one of these intervals or not belong to any of them, but it can not belong to more than one interval.

The following proposition will allow us obtain a characterization of the intervals $\left\{(k^s_{\alpha c}, k^s_{\beta c}),\ s = 1, ...\right\}$ using the concept of crossing point, that is a value of the capital-labor ratio k that verifies $f_\alpha(k) = f_\beta(k)$. The proposition says

Proposition 2: *Every crossing point is contained in an interval of* $\left\{(k^s_{\alpha c}, k^s_{\beta c}),\ s = 1, ...\right\}$. *Moreover, in every interval there is one and only one crossing point.*

Using the Proposition 2 we next show the order of the solutions of the system (2.4)-(2.5).

Proposition 3: *Assuming without lost of generality that* $k^1_\alpha < k^1_\beta$, *the solutions of the system (2.4)-(2.5) are ordered in the following form:*

$$k^1_{\alpha c} < k^1_{\beta c} < k^2_{\beta c} < k^2_{\alpha c} < k^3_{\alpha c} < ... \tag{2.6}$$

As we can observe, if the capital-labor ratio of a technology is he higher one in a pair of solutions, it will be the lower one in the next pair. As we will see in the

[3] Note that we can use the definition of $k_i(w)$ to obtain $k^s_{\beta c} = k_i(w^s_c)$.

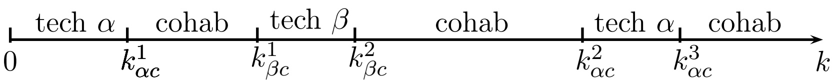

Figure 1: Different regimes depending on the value of k.

next theorem, it will be related to the technology used among these two intervals.

Finally, we state the following theorem, that shows the functional form of the aggregate production function:

Theorem 1: *When the solutions of the system (2.4)-(2.5) take the general form (2.6), the production function takes the following form:*

$$f(k) = \begin{cases} f_\alpha(k) & \text{if } k \in [k^{2s}_{\alpha c}, k^{2s+1}_{\alpha c}] \ , \\ w^s_c + R^s_c.k & \text{if } k \in [k^{2s+1}_{\alpha c}, k^{2s+1}_{\beta c}] \text{ or } k \in [k^{2s}_{\beta c}, k^{2s}_{\alpha c}] \ , \\ f_\beta(k) & \text{if } k \in [k^{2s+1}_{\beta c}, k^{2s+2}_{\beta c}], \end{cases}$$

for $s = 1, ...$ Then, when k is inside the intervals $\{(k^s_{\alpha c}, k^s_{\beta c}),\ s = 1, ...\}$ the economy is under cohabitation, and outside these intervals only one technology is used. Henceforth, the intervals $\{(k^s_{\alpha c}, k^s_{\beta c}),\ s = 1, ...\}$ will be called cohabitation intervals.

This theorem implies the existence of a sequence of cohabitation and non cohabitation intervals, indexed by the capital-labor ratio. This is shown schematically in the Figure 1. This model allows us to obtain endogenously a linear production function inside the cohabitation intervals.

The analysis of one cohabitation interval is sufficient to understand all possible outcomes of our model. In fact, as we have seen, all the cohabitation intervals are disconnected, allowing us to analyze the local properties of the production function (around the k of the economy) as if there were only one cohabitation interval. From now we will assume that there is only one cohabitation interval.[4] For convenience, we will focus on a single pair of solutions, that we denote $k_{\alpha c} \equiv k^s_{\alpha c}$ and $k_{\beta c} \equiv k^s_{\beta c}$, for a given $s > 0$. The same applies for $R_c \equiv R^s_c$ and $w_c \equiv w^s_c$. We can also assume, without lost of generality, that $k_{\alpha c} < k_{\beta c}$. This assumption says that the technology β is more capital-intensive than the technology α, because when k is high this is the

[4]If, for example, both functions are Cobb-Douglas with different factor shares, we have that the number of crossing points is exactly one.

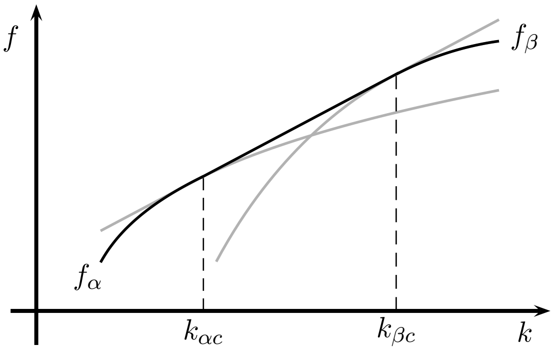

Figure 2: Production function around the cohabitation zone.

only technology used and in the cohabitation zone this technology requires a higher capital-labor ratio.

Theorem 1 implies the production function when the cohabitation interval is $(k_{\alpha c}, k_{\beta c})$ takes the following form:

$$f(k) = \begin{cases} f_\alpha(k) & \text{if } k \leq k_{\alpha c} \ , \\ w_c + R_c.k & \text{if } k \in (k_{\alpha c}, k_{\beta c}) \ , \\ f_\beta(k) & \text{if } k \geq k_{\beta c} \ . \end{cases} \tag{2.7}$$

The shape of the production function in the cohabitation zone is a straight segment that goes from $(k_{\alpha c}, f_\alpha(k_{\alpha c}))$ to $(k_{\beta c}, f_\beta(k_{\beta c}))$. Due the fact that $f'_\alpha(k_{\alpha c}) = f'_\beta(k_{\beta c}) = R_c$ we have that the function is smooth. This function is depicted in the Figure 2, where we can observe that if $k < k_{\alpha c}$ only the more labor-intensive technology is used (technology α), in the cohabitation interval the production function is linear and finally for $k > k_{\beta c}$ only the technology β is used.

An important result of this model is that the wage and the rental price in the range of cohabitation of both technologies do not depend on the capital-labor ratio. As we have seen, this is a consequence of the fact that the equations (2.4)-(2.5) do not depend on the capital-labor ratio k. If we display the wages and the rents of the capital as a function of k we observe, as we can see in the figure 3, that the behavior of these two variables depends significantly on whether k belongs to the cohabitation interval or not. Then we have that

$$w = \begin{cases} f_\alpha(k) - f'_\alpha(k).k & \text{if } k \leq k_{\alpha c} \ , \\ w_c & \text{if } k \in (k_{\alpha c}, k_{\beta c}) \ , \\ f_\beta(k) - f'_\beta(k).k & \text{if } k \geq k_{\beta c} \ , \end{cases} \quad \text{and} \quad R = \begin{cases} f'_\alpha(k) & \text{if } k \leq k_{\alpha c} \ , \\ R_c & \text{if } k \in (k_{\alpha c}, k_{\beta c}) \ , \\ f'_\beta(k) & \text{if } k \geq k_{\beta c} \ . \end{cases}$$

Both the wage and the capital rent are constant inside the cohabitation interval. In fact, inside this interval, changes in k produce a re-sharing of the factor endowments that preserve the values of the production factor rents.

> **Remark 1:** A direct consequence of the form of these two functions is that, when the rental price or the wage are hold exogenously to a given level, only one technology will be used. One of the most common effects of opening an economy to the international trade of capital factors use to be the convergence of the interest rate of the economy to the world interest rate. Given the curve of capital rents shown in the figure 3 (b) we see that, except in the case that the world interest rate coincides with the interest rate of cohabitation $R_c - \delta$, one of the two technologies will disappear in the long run. If, for example, the world interest rate is $r^* < R_c - \delta$, the capital becomes an abundant resource, and then only the capital intensive technology will be used. By the other hand, if $r^* > R_c - \delta$, then the capital becomes a scarce resource, and the only will be used the labor intensive technology.
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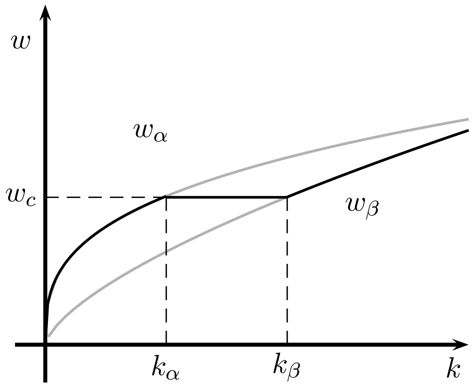
> This result has been recently studied in the international trade literature. Melitz (2003), Baldwin and Robert-Nicoud (2005) and Baldwin (2005), describe using some simple models the different behavior of industries that belong to the same sector when an economy opens its market to the international market, depending on the comparative advantage among the technologies used by the different firms once the trade is opened at international level. The results coincide with the results we have just presented, that is, before opening the economy two different technologies can cohabitate, but once the economy is opened to the international trade only the technology that has comparative advantage survives.

Inside the cohabitation interval variations on k shift the factor income shares of capital and labor, even when they are constant in the two single production functions (when they are Cobb-Douglas). The labor income share inside the cohabitation interval is given by

$$\frac{w_c.L}{Y} = \frac{w_c}{R_c.k + w_c} = \frac{1}{1 + \frac{R_c.k}{w_c}} \ ,$$

and the capital income share is

$$\frac{R_c.K}{Y} = \frac{R_c.k}{R_c.k + w_c} = \frac{1}{1 + \frac{w_c}{R_c.k}} \ .$$



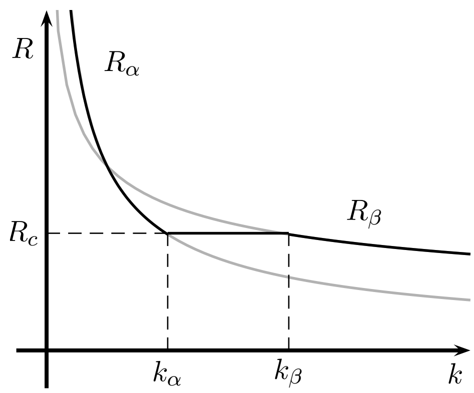


Figure 3: Graphic of w, in (a), and r, in (b), as functions of k.

Then, ceteris paribus, increases in k inside the cohabitation zone imply a higher participation of the capital in the national income and lower participation of the labor. In fact, the higher is k inside the cohabitation zone, the more used is the capital intensive technology, and then a higher share of the production is devoted to pay the capital rents.

> **Remark 2:** One possible extension of this model is to consider what happens when the number of available technologies is higher than two. This case seems to be more realistic, but, as we will now see, the different equilibria are equivalent to the model with only two technologies. In order to shed light into this case we could draw the relation of the rent of the capital in terms of the wage for all the available technologies, as in the figure 4, for 3 technologies. It is easy to prove that the equilibrium will be always in the outer line (in black). This is due to the fact that, for a given wage, the capital will choose the technology with larger rental price, and for a given rental price, the labor will choose the technology with larger wage. The subfigure (a) shows a situation where every technology will be used for some range of k, meanwhile the subfigure (b) shows a case where there is a technology that will never be used for any value of k.
>
> Cohabitation only occurs in the outer intersection points. These intersection points, in the general case, will be an intersection of two curves and no more (having a point where three curves cross is extremely rare). Then, in general, we will have two types of zones. When the economy lays in a point of the outer curve that belongs to only one technology then the model is the classical

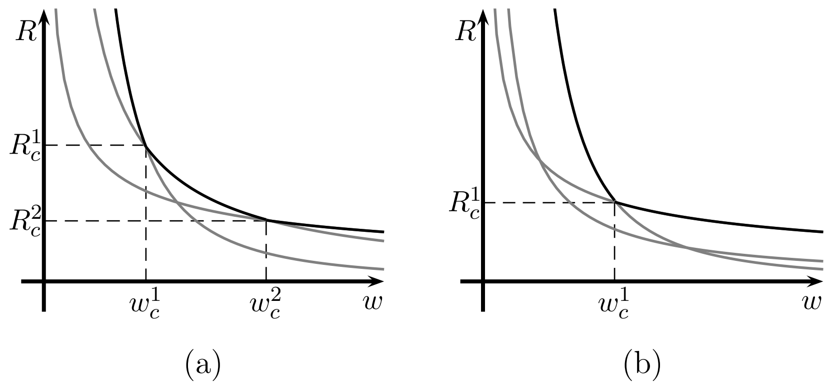


Figure 4: Graphic of rents of the factors when there are 3 available technologies.

Solow-Swan model, and only this technology is used. When the economy is in an intersection point between two technology curves then the analysis and results we have done hold. In this case, as we have seen, there is a range in the capital-labor ratio where the two technologies are used.

Remark 3: As we have just seen in Remark 2 it is clear that when we have two production factors, in our case labor and capital, in general only two technologies can be used at the same time. The question that now arise is whether an additional production factor allows the cohabitation of more than two technologies in our economy. Suppose now that we have N production factors and we ask if M technologies can be used simultaneously or not. If we use classical production functions with constant returns to scale only the production factor per unity of labor force used is important in the equations, so we have $(N-1).M$ unknown variables. The rents of every factor, including the labor, must be equal among all the technologies, that implies $M-1$ conditions for every production factor. Then, for N production factors and M technologies, the number of equations is given by $N.(M-1)$. The overall system establishes the following relations for the economy

$$\left.\begin{array}{ll}(M-1)\times N & \text{equations} \\ M\times(N-1) & \text{unknown variables}\end{array}\right\}(M-N) \;\; \text{deg. of freedom}$$

The condition for the system to be solvable implies that the number of degrees of freedom must be higher or equal than 0. This condition implies that in general the number of technologies simultaneously used must be less or equal than the number of production factors.

3 The economy

We now study the dynamics of an economy with two available technologies as those we have just presented. We allow the capital-labor ratio to move following the Solow dynamic equation, increasing with investment and decreasing with depreciation and the fertility rate. Since we assume that the markets clear at every moment of time, when the two technologies are used the equations (2.4) and (2.5) hold.

The analysis of the equilibrium is the same than in the classical model of Solow (1956) with the (non-neoclassical) production function (2.7). Given a depreciation rate δ, the capital accumulation is determined by the decisions of the people on the fertility n and the saving rate s, according to the following equation:

$$\dot{k} = s.f(k) - (\delta + n).k \ .$$

When the fertility and saving rate are constant and exogenous the steady-state condition implies a steady capital-labor ratio k^*. This is given by the following expression:

$$0 = \dot{k} = s.f(k^*) - (\delta + n).k^* \quad \Rightarrow \quad s = \frac{(\delta + n).k^*}{f(k^*)} \ . \tag{3.1}$$

In particular, the saving rates that leave the economy in the borders of the cohabitation interval are the following:

$$s_\alpha = \frac{(n+\delta).k_{\alpha c}}{f(k_{\alpha c})} = \frac{n+\delta}{w_c.k_{\alpha c}^{-1} + R_c} \quad \text{and} \quad s_\beta = \frac{(n+\delta).k_{\beta c}}{f(k_{\beta c})} = \frac{n+\delta}{w_c.k_{\beta c}^{-1} + R_c} \ . \tag{3.2}$$

Since $k_{\alpha c} < k_{\beta c}$ we have that $w_c.k_{\alpha c}^{-1} + R_c > w_c.k_{\beta c}^{-1} + R_c$, and then $s_\alpha < s_\beta$. Inside the cohabitation interval the steady state capital-labor ratio is given by the following formula:

$$k^* = \frac{s.w_c}{\delta + n - s.R_c} \ .$$

It is easy to check that when the economy is in an the steady state inside the cohabitation interval we have $s.R_c < \delta + n$, and then the denominator is positive and k^* is well defined. Every saving rate implies an equilibrium capital-labor ratio k^*, and this is in the cohabitation interval when $s \in (s_\alpha, s_\beta)$.

When the capital-labor ratio falls in the cohabitation interval, inside a neighborhood of the steady state the capital stock accumulation is given by

$$\dot{k} = s.f(k) - (\delta + n).k = s.w_c + \big(s.R_c - \delta - n\big).k \ .$$

Assuming $s = s^* \in [s_\alpha, s_\beta]$ constant, the previous equation is a linear differential equation that can be analytically solved. Then, if the initial value of the capital-labor ratio belongs to the cohabitation interval, the evolution of the economy will be given by the following expression:

$$\begin{aligned} k(t) &= e^{(s^*.R_c-\delta-n).t}.k_0 + \frac{s^*.w_c}{s^*.R_c-\delta-n}\cdot(1-e^{(s^*.R_c-\delta-n).t}) \\ &= e^{(s^*.R_c-\delta-n).t}.k_0 + (1-e^{(s^*.R_c-\delta-n).t}).k^* \ . \end{aligned}$$

The condition $s^*.R_c < \delta + n$, that as we have seen is a necessary condition to fall in the cohabitation interval, ensures that the steady state is stable.

Let's now inquire on how changes in the productivity of the different factors affect to technological change. We assume productivity shocks and then we calculate the new cohabitation interval. Denote $k^{\text{in}}_{\alpha c}$ and $k^{\text{in}}_{\beta c}$ the initial boundary values of the cohabitation zone. Assume that we can decompose the increase in the productivity using two parameters: the Harrod neutral (labor augmenting) parameter, denoted by B, and the Solow neutral (capital augmenting) parameter, denoted by C. We suppose that $B, C \geq 1$. The parameters B and C can be interpreted as increases in productivities of the factors.

Assume a productivity shock parameterized by B and C. We denote the production function after the increase as F^t_i, with $i = \alpha, \beta$. Then, using the constant returns to scale, we can write the following expression

$$F^t_i(K_i, L_i) = F_i(C.K_i, B.L_i) = B.L_i.f_i\left(\frac{C.k_i}{B}\right) \ ,$$

with $i = \alpha, \beta$. Let's define $\hat{k}_i = \frac{C.k_i}{B}$, for $i = \alpha, \beta$. Now, the conditions of the competitive market when the economy is under cohabitation are given by

$$B.\big(f_\alpha(\hat{k}_\alpha) - \hat{k}_\alpha.f'_\alpha(\hat{k}_\alpha)\big) = B.\big(f_\beta(\hat{k}_\beta) - \hat{k}_\beta.f'_\beta(\hat{k}_\beta)\big) \ , \tag{3.3}$$

$$B.f'_\alpha\left(\hat{k}_\alpha\right) = B.f'_\beta\left(\hat{k}_\beta\right) \ . \tag{3.4}$$

The equations (3.3) and (3.4) for $\hat{k}_\alpha$ and $\hat{k}_\beta$ are exactly the same than the equations (2.4) and (2.5) for k_α and k_β. Then, the cohabitation interval is bounded by $\hat{k}_{\alpha c} = k^{\text{in}}_{\alpha c}$ and $\hat{k}_{\beta c} = k^{\text{in}}_{\beta c}$, and the economy will be under cohabitation when $\hat{k} \in (k^{\text{in}}_{\alpha c}, k^{\text{in}}_{\beta c})$.

The interval of cohabitation, in terms of the capital-labor ratio, is given by

$$(k_{\alpha c}, k_{\beta c}) = \left(\frac{\hat{k}_{\alpha c}.B}{C}, \frac{\hat{k}_{\beta c}.B}{C}\right) = \frac{B}{C}\cdot\left(k^{\text{in}}_{\alpha c}, k^{\text{in}}_{\beta c}\right) \ .$$

We observe that the new interval of cohabitation changes with respect the initial interval. Moreover, the influence of the increase of the productivity on the interval of cohabitation depends on the kind of increase of productivity, which can be Solow neutral or Harrod neutral.

The Harrod neutral (labor augmenting) part of the technological advance shifts the cohabitation interval to higher capital-labor ratios, and this interval becomes broader. When the productivity of the labor rises, the labor intensive technology becomes cheaper than before, and then transition to the capital intensive production function takes place in high capital-labor ratios. The Solow neutral (capital augmenting) part of the technological advance shifts the cohabitation interval to lower capital labor ratios, and the interval becomes narrower. When capital becomes more productive, the capital intensive technology becomes cheaper than before, and the transition to the capital intensive production function takes place in lower capital-labor ratios.

The increase on the productivity, both when is Harrod neutral and when is Solow neutral, produces an increase of the steady capital-labor and the income per capita. This is also true when there is no cohabitation. Nevertheless, when an increase of the productivity occurs, it not always implies a shift toward the capital intensive technology. As we have seen, on the one hand, a labor augmenting increase of the productivity can shift the cohabitation interval to higher capital-labor ratios, increasing the use of the labor-intensive technology. On the other hand, the increase in the productivity increases the steady state capital-labor ratio of the economy. In order to analyze these two effects let's consider the equation (3.1) with the increased productivity production function f^t, that turns to be

$$s = \frac{(\delta+n).k^*}{f^t(k^*)} = \frac{(\delta+n)\cdot\frac{B}{C}\cdot\hat{k}^*}{B.f(\hat{k}^*)} = \frac{(\delta+n)\cdot\hat{k}^*}{C.f(\hat{k}^*)}$$

Because the economy is under cohabitation when $\hat{k} \in (k^{\text{in}}_{\alpha c}, k^{\text{in}}_{\beta c})$, the cohabitation interval for the saving rates is $s \in \frac{1}{C}(s_\alpha, s_\beta)$, with s_α and s_β defined in (3.2). Then the effect of a shift the economy to the labor intensive technology when the advance is Harrod-neutral is counterbalanced by the increasing in the capital-labor ratio due to the increasing of the production. When the innovation is Solow neutral the economy moves always toward the capital intensive technology.

4 Migration of factors

Nowadays immigration plays an important role in the study of the economies of many countries. Migrations of factors alter the endowments of an economy, and then the use of the technologies. So, it is interesting to analyze the role of migrations in the technological change and the consequences for the important variables of the economy.

Let's then introduce migrations of factors in our model. The introduction of migrations of production factors allows us to study their effect on the technological change. In fact, in the model we have just developed, the shares of endowments used in each technology depend on the endowments of the economy. Then, modifications of these endowments produced by migrations can cause technological change. We maintain the assumption that the economy is closed to international trade, but now it can receive instantaneous migration shocks of production factors that alter its endowments. In this section we study how these shocks affect technological change and income.

We first analyze how do small immigration shocks affect the income of the residents of a country. It is an important issue, because the acceptation of migrations by the residents in a country depends on how migrations affect their incomes. In order to do this, we introduce a difference among residents in our country, given by the number of assets they have. Because there are no international capital flows, the total number of assets in the economy coincides with the total capital.

Assume that every individual supplies inelastically an unit of labor. Consider an individual that holds a_i assets. Then, if the capital-ratio of the economy is k (that coincides with the average number of assets, given by $\overline{a_i}$), the income y_i of this individual is given by

$$y_i = w + R.a_i = y(k) + (a_i - k).y'(k) \ ,$$

where $y(k)$ is the average income per capita of the economy. Suppose now that there is an immigration shock of labor, changing the total labor force from L to $L + \mathrm{d}L$, and then the capital-labor ratio is also altered. The variation of the capital-labor ratio is given by[5]

$$k + \mathrm{d}k = \frac{K}{L + \mathrm{d}L} = k.\left(1 - \frac{\mathrm{d}L}{L}\right) + O\left(\left(\frac{\mathrm{d}L}{L}\right)^2\right) \ .$$

[5] $O(\mathrm{d}x)$ denote terms that go to 0 when $\mathrm{d}x \to 0$ like $\mathrm{d}x$.

The new average income of the residents can be calculated in the following form:

$$\begin{aligned}\overline{y_i}(k+\mathrm{d}k) &= \overline{y(k+\mathrm{d}k)+(a_i-k-\mathrm{d}k).y'(k+\mathrm{d}k)} \\ &= y(k+\mathrm{d}k)+(\overline{a_i}-k-\mathrm{d}k).y'(k+\mathrm{d}k) \\ &= y(k)+O((\mathrm{d}k)^2)\ ,\end{aligned}$$

where the upper bar indicates the average among the former residents. As we can see, small immigrations of new production factors do not alter, at first order, the aggregated income of the domestic residents[6]. This effect arises because the new additional workers and capital rents are paid their marginal production, that is, the additional product they produce. This fact is noted by Borjas (1999) and Ben-Gad (2004), who show that the increase of the labor force and the subsequent decrease of the capital-labor ratio shift up the capital rents of native-owed capital, that slightly surpasses the decrease of the wages.

Although the average domestic income is not altered, the migration factors produces a redistribution of this income among the residents. The factor that is comparatively increased, in our case the labor force, becomes more abundant, and then loses rent per unity of factor, and the factor comparatively decreased, in our case the capital, becomes more scarce, and then increases rent per unity of factor. For an individual that owns an amount of assets equal to a_i the variation of his or her income is given by

$$\frac{\partial y_i}{\partial k} = (a_i - k).y''(k)\ .$$

Then, for general neoclassical production functions where $y''(\cdot) < 0$, the redistribution affects all the people who do not own exactly the average capital of the economy. If, for example, there is an immigration shock of labor force, people who owns less assets than the average loses income, and the income of people who owns more assets than the average increases. This fact increases the income inequality. On the one hand the income of people who owns a low number of assets, income that was low before the immigration, decreases, becoming poorer. By the other hand, people whose income was high before the immigration because they had a high number of assets, increase their income. Therefore, the immigration of new labor force tends to reinforce income inequalities. The increasing in the labor force lowers the wages,

[6]In fact, when $\mathrm{d}L \ll L$ the quadratic terms in $\mathrm{d}L/L$ are negligible. If, for example the immigrated labor force is the 5% of the economy, the wages will be lowered roughly a 5%, but the variation of the average income of the residents will be a small 0.25%.

that is the part of the income that is common to all the agents, and reinforces the capital returns, that is the source of difference among the incomes of the agents.

As we have seen, when there are two available technologies and the economy is under cohabitation, the production function takes the form (2.7), that is locally linear. In this case, in cohabitation, $y''(\cdot) = 0$, and then there is no redistribution of income among the residents when there is a migration of production factors. This is a consequence of the constancy of the wages and the capital returns inside the cohabitation zone, which imply that the income does not change when k changes. Then, it seems that when the economy is under cohabitation, the individuals are indifferent to the migration of factors, at least for personal income reasons, and less income inequality will be caused by these migrations.

Nevertheless, in an economy under cohabitation another important effect may arise. When new factors immigrate to the economy, they are shared up among the technologies depending on their capability to absorb them. The question we now inquire is whether the factors that the economy had before the migration are forced to move between the technologies or not. Suppose a immigration shock of labor force in our the economy. In this case we can identify two effects that move in opposite directions: the production level effect and the technology adoption effect. The first effect implies that both technologies will use more labor, as a result of the increase in the total amount of labor. The technology adoption effect implies that, a since labor now is cheaper than before, the labor-intensive technology will be more used, using more labor than before. However the capital-intensive technology will be less used, using less labor than before. In order to inquire which effect dominates, let's consider two scenarios, depending on the mobility of the factors across the technologies.

Assume first that the factors are instantaneously and costlessly adaptable from one technology to another. Assume also that the economy is under cohabitation when the labor force is augmented from L to $L + \mathrm{d}L$. If this shock takes place in a short period of time, we can assume that the endowment of capital in our economy remains constant in his original value K. The capital-labor ratio shifts, and the factors endowed in the two technologies move across technologies in order to preserve the equality of the wages and the rental prices between the two technologies. On the one hand, the variations of the capital endowed in the two technologies with respect the variation of the population are given by the following results:

$$\frac{\mathrm{d}K_\alpha}{\mathrm{d}L} = -\frac{\mathrm{d}K_\beta}{\mathrm{d}L} = \frac{k_{\alpha c}.k_{\beta c}}{k_{\beta c} - k_{\alpha c}} > 0 \ .$$

We observe that the immigration of new labor induces a movement of capital from the capital intensive technology to the labor intensive technology. Because the capital is assumed to be constant, the increase of capital in the labor intensive sector must be equal to the decrease of the capital intensive sector. On the other hand, the changes in the labor force endowed in the two technologies are given by

$$\frac{\mathrm{d}L_\alpha}{\mathrm{d}L} = \frac{k_{\beta c}}{k_{\beta c} - k_{\alpha c}} > 1 \quad \text{and} \quad \frac{\mathrm{d}L_\beta}{\mathrm{d}L} = -\frac{k_{\alpha c}}{k_{\beta c} - k_{\alpha c}} < 0 \ .$$

From the previous results we observe that the technology adoption effect dominates, because there is a decrease in the labor force and the production in the capital intensive sector. This implies that the relative increase of the labor force endowed in the labor intensive technology with respect the increase of the total labor force is greater than 1. All the new labor force is endowed in this technology and also part of the labor force that comes from the decrease of labor force of the capital intensive technology.

We observe that, even if all the immigrants are endowed in the labor intensive technology many domestic residents of the country have to change the technology where they work, in order to maintain the equalization capital rents and wages among the two technologies. Moreover, some capital have to move from the capital intensive technology to the labor intensive technology. It means that, for every new immigrant (suppose that she chooses to work in the labor intensive technology) $k_{\alpha c}/(k_{\beta c} - k_{\alpha c})$ domestic workers have to move from the capital intensive technology to the labor intensive technology.

In order to understand the previous results let's introduce a second scenario. In this scenario, we assume that there is no mobility of capital among the two technologies. We assume that the capital is produced to be endowed in a concrete technology, and then it can not be use in the other one. Then, the replacement of the two technologies along the time occurs only through the depreciation of the capital and the production of new capital that can be adapted to every technology.

We consider a immigration shock of labor force, denoted by $\mathrm{d}L$, is shared among the two technologies in the quantities, $\mathrm{d}L_\alpha$ and $\mathrm{d}L_\beta$, with $\mathrm{d}L_\alpha + \mathrm{d}L_\beta = \mathrm{d}L$. We assume that the arrival of the immigration and its introduction to the endowed labor force occur in a relatively short period of time, and then we also assume that the capital endowed in the two technologies does not vary during this short adaption period. Then, the new capital-labor ratios for the two technologies are given by

$$k_\alpha = \frac{K_\alpha}{L_\alpha + \mathrm{d}L_\alpha} \quad \text{and} \quad k_\beta = \frac{K_\beta}{L_\beta + \mathrm{d}L_\beta} \ .$$

When new immigrant workers arrive to our economy, they choose to work in the technology with higher wage. Then, the wages of the two technologies are equal at every moment. The condition of equal wages in the two technologies is given by

$$f_\alpha(k_\alpha) - k_\alpha . f'_\alpha(k_\alpha) = f_\beta(k_\beta) - k_\beta . f'_\beta(k_\beta) \ . \tag{4.1}$$

Using the assumption that the economy before the immigration shock was under cohabitation, and thus verifying the equations (2.4) and (2.5), and assuming that the immigrated labor force is sufficiently smaller than the actual labor force of the economy, $\mathrm{d}L \ll L$, this equation can be reduced to a first order equation. This equation turns to be

$$k_{\alpha c}^2 \cdot \frac{\mathrm{d}L_\alpha}{L_\alpha} \cdot f''_\alpha(k_{\alpha c}) = k_{\beta c}^2 \cdot \frac{\mathrm{d}L_\beta}{L_\beta} \cdot f''_\beta(k_{\beta c}) \ .$$

The variations of the sharing of the labor force are given by:

$$\frac{\mathrm{d}L_i}{\mathrm{d}L} = \frac{k_{jc}^2 \cdot \frac{f''_j(k_{jc})}{L_j}}{k_{\alpha c}^2 \cdot \frac{f''_\alpha(k_{\alpha c})}{L_\alpha} + k_{\beta c}^2 \cdot \frac{f''_\beta(k_{\beta c})}{L_\beta}} > 0 \ ,$$

with $j \neq i$. Note that in this second scenario the labor force used in both technologies increases, and then the production level effect dominates.

The equalization of the wages among the technologies does not ensure that the returns of the capital will be equal. In fact, the additional labor force to both technologies rises the returns of the capital rents in both sectors, but these increases can be different, due to the non mobility of the capital across the technologies. The immigration affects the rental prices of the two technologies in the following way>:

$$\frac{\mathrm{d}R_i}{\mathrm{d}L} = \frac{\mathrm{d}f'_i(k_i)}{\mathrm{d}L} = \frac{\partial f'_i(k_i)}{\partial k_i}\frac{\mathrm{d}k_i}{\mathrm{d}L} = \frac{1}{k_{ic}} \cdot \frac{\overbrace{k_{\alpha c}^2 \cdot \frac{f''_\alpha(k_{\alpha c})}{L_\alpha} \cdot k_{\beta c}^2 \cdot \frac{f''_\beta(k_{\beta c})}{L_\beta}}^{G(k_{\alpha c}, k_{\beta c})}}{k_{\alpha c}^2 \cdot \frac{f''_\alpha(k_{\alpha c})}{L_\alpha} + k_{\beta c}^2 \cdot \frac{f''_\beta(k_{\beta c})}{L_\beta}},$$

with $j \neq i$. As we see, capital returns increase in the two technologies. This comes from the fact that the labor force increases in both technologies, and then all the capital increases its marginal productivity in both technologies. We observe that in every technology this increase depends both on a function $G(k_{\alpha c}, k_{\beta c})$, that does not depend on the technology being considered, and also on the inverse of the capital-labor ratio of this technology. Then, the increase on the rental price of the technology α (with lower capital-labor ratio in cohabitation) is larger than the

	Cohab. scen. 1		Cohab. scen. 2		Non cohab.	
	α	β	α	β	α	β
w_i	0	0	-1.92%	-1.92%	-1.61%	-3.20%
R_i	0	0	+3.97%	+0.98%	+3.31%	+1.64%
L_i/L	+6.67%	-1.67%	+4.01%	+0.99%	+5%	+ 5%
K_i/K	+3.33%	-3.33%	0	0	0	0
Y_i/Y	+5.00%	-2.50%	+1.99%	+0.49%	+3.31%	+1.64%
y_i	-2.38%		-2.40%		-1.61%	-3.20%

Table 1: Comparison of the effects of a 5% immigration shock in the different scenarios and when there is no cohabitation.

increase of rental price of the technology β, and the new capital produced in the economy will be adapted to the labor intensive technology.

In order to clarify the results that we have obtained we consider a numerical example. Assume that the two production functions are Cobb-Douglas, $Y_i = A_i.K_i^{\sigma_i}.L_i^{1-\sigma_i}$, with $i = \alpha, \beta$. We assume that the output elasticities of capital take the values with $\sigma_\alpha = 1/3$, $\sigma_\beta = 2/3$, respectively. We also assume that they have the same total factor productivity. In this case the cohabitation interval is $k \in [\frac{1}{2}, 2]$. Assume that s is such that $k^* = 1$.

Table 1 shows the effects to different variables of an immigration shock of 5%. This table shows the effects in the two scenarios that we have just presented and the effects when only one technology is available. We that the wage and the capital rents are not altered in the first scenario, because the economy remains in cohabitation. In the second scenario we observe that the capital rent increases more in the labor-intensive technology, as we predicted before. The labor force is shared as we predicted in the two scenarios, with variations with different sign in the first scenario and positive variations in the second scenario. The same signs are observed for the variations of the output and income per capita. The resulting income per capita is slightly lower in the second scenario than in the first one, fact that indicates less capacity of adjustment of the factors, in this case the capital. It is interesting to note that cohabitation results are all intermediate values between the values obtained when only one technology is used.

5 Conclusions

In this paper we have studied an economy where two technologies are available. The first result we have obtained is that cohabitation occurs for certain regions of the capital-labor ratio of the economy, and there are steady states with and without cohabitation.

All this paper is based on general neoclassical production functions, and then the results obtained are independent of their particular functional form. Cohabitation of two technologies is a general case when there are two production factors, as we have noted in the Remark 1. We have shown that the results hold even if there were many technologies available. The analysis can be significantly reduced to the study of only two locally different situations: cohabitation and non cohabitation.

We have shown that when a factor payment is fixed exogenously only one technology will subsist this process. This result is in accordance to many international trade literature, where the process of opening to international markets may imply convergence in some factor payments, specially the capital. In fact, when the economy is opened to international trade and the interest rate is held exogenously, one technology becomes more profitable for the firms, and then there is technological change. While the economy is closed cohabitation occurs in a wide subset of the parameters space, but once it is opened cohabitation only occur in a single point.

This model allows us to make precise predictions about the shape of the production function in certain regions of the capital-labor ratio. Regardless the form of the production functions of the technologies available, the joint production function is linear when there is cohabitation. This fact implies constant wages and capital rents, producing a region with high stability on people's incomes, because they do not depend on the capital-labor ratio of the economy. Moreover, the factor shares vary along the cohabitation interval depending on the capital-labor ratio. This endogenous change of the factor shares occurs even if the production functions of both technologies are Cobb-Douglas. This fact can be used in the future to treat the observed evolution of the factor income shares.

The analysis of the effects of migrations of factors are significantly different when more than one technology than when only one technology is available. The different scenarios that we have considered allow us to analyze different effects on the shares of the factors and their payments. The differences in the effects of migrations depend crucially on the mobility across technologies of the production factors. Interestingly, migrations of factors affect differently the rents of the factors employed in the different technologies, and then native factors are forced to change the technology where they are used.

The simplicity of this model helps us to analyze the process better, and allows future generalizations improve our include a wide range of effects. As we noted in the Remark 2, when there are more than two production factors more than two technologies can be used simultaneously, possibly implying more general results. Moreover, introducing new goods can be another interesting way to compare our results observed with the empirical data.

Summarizing, this paper generalizes the Solow model allowing the coexistence of different technologies. This simple generalization allows the introduction of heterogeneity of firms and analyze the effects of opening the economy to international trade. Moreover, it allows us to explain how productivity changes and migration of factors affect both endogenous technological change and income.

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Appendix

We now give some mathematical results that will help us to analyze the behavior of the solutions of the equation system (2.4)-(2.5) and their implications for the cohabitation. These results will help us to understand better the implications of our study and will allow us to simplify the following analysis.

In order to study what conditions allow cohabitation to occur lets study first the properties of the set of solutions of the system (2.4)-(2.5). We begin comparing the two technological production functions and their derivatives when k takes values in these solutions. Consider a pair of solutions $\{k^j_{\alpha c}, k^j_{\beta c}\}$, for a given value of $j = 1, \ldots$. We first compare, using the concavity of the production functions, the derivatives of the production functions inside the interval generated by this pair solutions. Noting that the derivatives of the two production functions are decreasing functions and using (2.5), it is easy to prove that

$$k^j_{ic} < k^j_{i'c} \qquad \Rightarrow \qquad f'_i(k) < f'_{i'}(k), \quad \forall k \in (k^j_{ic}, k^j_{i'c}), \quad \text{for } i, i' \in \{\alpha, \beta\} \; . \tag{5.1}$$

As we can see the derivatives of the production functions do not cross inside the intervals generated by the solutions. Let's now compare the production functions evaluated at k^j_{ic}:

$$\begin{aligned} f_i(k^j_{ic}) - f_{i'}(k^j_{ic}) &= f_{i'}(k^j_{i'c}) - f_{i'}(k^j_{ic}) + (k^j_{ic} - k^j_{i'c}).R^j_c \\ &= (k^j_{ic} - k^j_{i'c}).R^j_c.\left(1 - \underbrace{\frac{1}{f'_{i'}(k^j_{i'c})} \cdot \frac{f_{i'}(k^j_{ic}) - f_{i'}(k^j_{i'c})}{k^j_{ic} - k^j_{i'c}}}_{m}\right) . \end{aligned}$$

Now, imposing another time concavity on the production functions, we have that m is lower than 1, and then $f_i(k^j_{ic}) > f_{i'}(k^j_{ic})$. Then, using that the neoclassical production functions are increasing functions of the capital-labor ratios, we have the following result:

$$k^j_{ic} < k^j_{i'c} \qquad \Rightarrow \qquad f_{i'}(k^j_{ic}) < f_i(k^j_{ic}) < f_i(k^j_{i'c}) < f_{i'}(k^j_{i'c}) \; . \tag{5.2}$$

This ordering of the values of the production functions will be useful to obtain properties of the solutions of our problem.

Let's now introduce a technical result, given in form of lemma, that will help us to prove the propositions and the theorems we have used in the paper:

Lemma 1: *For every pair $\{k^j_{\alpha c}, k^j_{\beta c}\}$ solution of (2.4)-(2.5) there is a straight line that is tangent to the production function of the technology α in the point $k^j_{\alpha c}$*

and tangent to the production function of the technology β at $k^j_{\beta c}$. This line is given by

$$l^j(k) \equiv w^j_c + R^j_c.k = y(k, k^j_{\alpha c}, k^j_{\beta c}) \ . \tag{5.3}$$

where $y(k, k^j_{\alpha c}, k^j_{\beta c})$ is defined in (2.2).

Proof: Let's first check the equality $l^j(k) = y(k, k^j_{\alpha c}, k^j_{\beta c})$. This equality can be prooved using $f_i(k^j_{ic}) = w^j_c + R^j_c.k^j_{ic}$, that implies

$$\begin{aligned} y(k, k^j_{\alpha c}, k^j_{\beta c}) &= \frac{k^j_{\beta c} - k}{k^j_{\beta c} - k^j_{\alpha c}} \cdot (w^j_c + R^j_c.k^j_{\alpha c}) + \frac{k - k^j_{\alpha c}}{k^j_{\beta c} - k^j_{\alpha c}} \cdot (w^j_c + R^j_c.k^j_{\beta c}) \\ &= w^j_c + R^j_c.k \ . \end{aligned}$$

The tangency comes from the fact that the straight line have only one point in common with each of the two curves[7] and the same slope in these points[8]. □

This lemma says that if the economy is under cohabitation, the production function must take a linear form. The Figure 2 shows the graphical implication of this result.

The next proposition states an interesting property of the intervals generated by the pairs of solutions of the system (2.4)-(2.5). This property is important because allows to study locally only two types of capital-labor ratios, those which are contained in one of these intervals and those which do not belong to any of them.

Proposition 1: *The intervals generated by the pairs of solutions of (2.4)-(2.5) do not intersect.*

Proof: Suppose that the pairs $\{k^i_{\alpha c}, k^i_{\beta c}\}$ and $\{k^j_{\alpha c}, k^j_{\beta c}\}$, with $i \neq j$, are two different solutions of the system (2.4)-(2.5), and without loosing generality that $k^i_{\alpha c} < k^i_{\beta c}$. From (5.1) we have that $f'_\alpha(k) > f'_\beta(k)$ for $k \in (k^i_{\alpha c}, k^i_{\beta c})$. This condition ensures that is not possible that both $k^j_{\alpha c}$ and $k^j_{\beta c}$ belong to $(k^i_{\alpha c}, k^i_{\beta c})$ and verify $f'_\alpha(k^j_{\alpha c}) = f'_\beta(k^j_{\beta c})$.

[7] Because the production functions are neoclassical the rents of the factors absorb all the product. Then, we have that $f_j(k^i_{jc}) = w^i_c + R^j_c.k^j_{jc}$, being $j = 1, 2$ the technology considered.

[8] The property the productions functions have the same slope in the solutions of (2.4)-(2.5) comes from the fact that the interest rates, that determine the slope of the productions functions, are the same in these points.

Assume first $k^j_{\alpha c} > k^j_{\beta c}$ and $(k^j_{\beta c}, k^j_{\alpha c}) \cap (k^j_{\alpha c}, k^j_{\beta c}) \neq \emptyset$. Inside the intersection we have $f'_\alpha(\cdot) < f'_\beta(\cdot)$ by using (5.1) with the condition $k^i_{\alpha c} < k^i_{\beta c}$. Moreover $f'_\alpha(\cdot) < f'_\beta(\cdot)$ by using (5.1) with the condition $k^j_{\alpha c} > k^j_{\beta c}$, which is a contradiction. Then, assume that $k^j_{\alpha c} < k^j_{\beta c}$. We demonstrate the case when $k^j_{\alpha c} \in (k^i_{\alpha c}, k^i_{\beta c})$ and $k^j_{\beta c} > k^i_{\beta c}$, and the other case is analogous. Due to the concavity of the production function f_α we have that $f_\alpha(k^j_{\alpha c}) < l^i(k^j_{\alpha c})$ and $R^j_c = f'_\alpha(k^j_{\alpha c}) < R^i_c$, and then the straight line (5.3) corresponding to the pair j, $l^j(k)$ is strictly lower than $l^i(k)$ for $k > k^j_{\alpha c}$. Then, in particular, it crosses the production function $f_\beta(\cdot)$ inside the interval $(k^j_{\alpha c}, k^i_{\beta c})$. The property that a convex curve that crosses a line in one point can not be tangent to this line at any other point ensures that does not exist any point $k^i_{\beta c}$ that is tangent to the curve. Nevertheless, Lemma 1 says that it is a necessary condition for the existence of the pair $\{k^j_{\alpha c}, k^j_{\beta c}\}$. □

Lets now characterize the cohabitation intervals with the crossing points between the two production functions:

Proposition 2: *If there exists k_T such that $f_i(k_T) = f_j(k_T)$ and $f'_i(k_T) > f'_j(k_T)$, then exists a cohabitation zone that contains k_T, with i and j different belonging to $\{\alpha, \beta\}$.*[9] *Moreover, inside every cohabitation interval there is one and only one crossing point.*

Proof: Suppose that we have an economy with L unities of labor force and K unities of capital, with the production functions verifying $f_i(k_T) = f_j(k_T)$, being $k_T = K/L$. Now, suppose that we endow the same labor force $L/2$ to every technology, $\frac{K+dK}{2}$ unities of capital to the technology i and $\frac{K-dK}{2}$ unities of capital to the technology j, with $dk > 0$. Then we have that the production becomes

$$\begin{aligned}\frac{f_i(k_T+dk)+f_j(k_T-dk)}{2} &= \frac{f_i(k_T)+f'_i(k_T).dk+f_j(k_T)-f'_j(k_T).dk}{2}\\ &= \underbrace{\frac{f_i(k_T)+f_j(k_T)}{2}}_{=f_i(k_T)=f_j(k_T)}+\underbrace{\frac{f'_i(k_T)-f'_j(k_T)}{2}}_{>0}\cdot dk\end{aligned}$$

Since this production is larger than when only the technology i is used (and analogously for the technology j), to use only one technology is not optimal. Then, using only one technology when $k = k_T$ is not a competitive equilibrium. The fact

[9] The condition $f'_i(k_T) > f'_j(k_T)$ is not necessary except in the case where the two curves are tangent in some point, i.e., $f_i(k_T) = f_j(k_T)$ and $f'_i(k_T) = f'_j(k_T)$. In this case there is the cohabitation zone reduces the point k_T.

that in one cohabitation zone there is at least one crossing point comes from the relation (5.2). Suppose now two crossing points, k_T and k'_T, with $k_T < k'_T$, without any crossing point between them. Then, if $f'_i(k_T) > f'_j(k_T)$ (an then $f_i(k)$ is locally higher than f_j for $k > k_T$) we must have $f'_i(k'_T) < f'_j(k'_T)$ (an then $f_i(k)$ is locally higher than f_j for $k < k'_T$). In this case we can write

$$f'_i(k_{\alpha c}) > f'_i(k_T) > f'_j(k_T) > f'_j(k'_T) > f'_j(k_{\beta c})$$

It is in contradiction with the condition $f'_i(k_{\alpha c}) = f'_j(k_{\beta c})$. $\square$

The next proposition, using the proposition we have just demonstrated, allows us to order the solutions of the system of equations (2.4)-(2.5) in a strictly increasing order:

Proposition 3: *Assuming without loosing generality that $k^1_\alpha < k^1_\beta$, the solutions of the system (2.4)-(2.5) are ordered in the following form:*

$$k^1_{\alpha c} < k^1_{\beta c} < k^2_{\beta c} < k^2_{\alpha c} < k^3_{\alpha c} < ...$$

Proof: Consider two pairs of solutions of the equations system (2.4)-(2.5), $\{k^j_{\alpha c}, k^j_{\beta c}\}$ and $\{k^{j+1}_{\alpha c}, k^{j+1}_{\beta c}\}$, without any pair of solutions among them, with $k^j_{\alpha c} < k^j_{\beta c}$. Suppose that $k^{j+1}_{\alpha c} < k^{j+1}_{\beta c}$. The equation (5.2) implies that $f_\beta(k^j_{\beta c}) > f_\alpha(k^j_{\beta c})$ and Proposition 2 establishes that there are no crossing points between the two pairs of solutions. Then $f_\beta(k) > f_\alpha(k)$ for k betwen the two intervals, and in particular we have $f_\beta(k^{j+1}_{\alpha c}) < f_\alpha(k^{j+1}_{\alpha c})$. This fact is in contradiction with the property (5.2). Then $k^{j+1}_{\alpha c} > k^{j+1}_{\beta c}$. $\square$

Finally, the next theorem gives us the explicit form of the production function, and establishes the regions of the capital-labor ratio k where the economy is under cohabitation.

Theorem 1: *When the solutions of the system (2.4)-(2.5) take the general form (2.6) the production function takes the following form*

$$f(k) = \begin{cases} f_\alpha(k) & \textit{if } k \in [k^{2j}_{\alpha c}, k^{2j+1}_{\alpha c}] \ , \\ w^j_c + R^j_c.k & \textit{if } k \in (k^j_{\alpha c}, k^1_{\beta c}) \ , \\ f_\beta(k) & \textit{if } k \in [k^{2j+1}_{\beta c}, k^{2j+2}_{\beta c}], \end{cases}$$

for $j = 1, ...$ *Then, cohabitation occurs only and always inside the cohabitation intervals* $(k^j_{\alpha c}, k^1_{\beta c})$, *for* $j = 1, 2, ...$

Proof: Inside the interval generated by each solution, $k \in (k^j_{\alpha c}, k^j_{\beta c})$, the cohabitation is possible, and allows a production equal to $y(k, k^j_{\alpha c}, k^j_{\beta c})$, defined in (2.2). This is, as is stated in Lemma 1, a straight line (5.3) tangent to the two curves. Cause the concavity of the production functions, the production in cohabitation is higher than the production functions of the two technologies in this interval. The outcome of the free market implies the maximum production, that is, the economy will be under cohabitation, with the production function equal to $w^i_c + R^i_c.k$.

For $k \in [k^{2j}_{\alpha c}, k^{2j+1}_{\alpha c}]$ we know that any kind of cohabitation in this interval is not a competitive equilibrium. Then only one technology will be used in this interval, because the market can not provide a cohabitation among the two technologies. The equation (5.2) ensures that f_α is higher than f_β in the extremes of the interval, while Proposition 2 ensures that there is no crossing point among these production functions inside the interval. Then, only the technology α will be used inside this interval, because its production is higher. The same applies for $k \in [k^{2j+1}_{\beta c}, k^{2j+2}_{\beta c}]$, being the technology β the only used in this interval. $\square$